

Continuous-time Markov Decision Processes with Finite-horizon Expected Total Cost Criteria

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Abstract

This paper deals with the unconstrained and constrained cases for continuous-time Markov decision processes under the finite-horizon expected total cost criterion. The state space is denumerable and the transition and cost rates are allowed to be unbounded from above and from below. We give conditions for the existence of optimal policies in the class of all randomized history-dependent policies. For the unconstrained case, using the analogue of the forward Kolmogorov equation in the form of conditional expectation, we show that the finite-horizon optimal value function is the unique solution to the optimality equation and obtain the existence of an optimal deterministic Markov policy. For the constrained case, employing the technique of occupation measures, we first give an equivalent characterization of the occupation measures, and derive that for each occupation measure generated by a randomized history-dependent policy, there exists an occupation measure generated by a randomized Markov policy equal to it. Then using the compactness and convexity of the set of all occupation measures, we obtain the existence of a constrained-optimal randomized Markov policy. Moreover, the constrained optimization problem is reformulated as a linear program, and the strong duality between the linear program and its dual program is established. Finally, a controlled birth and death system is used to illustrate our main results.

Keywords. Continuous-time Markov decision processes; finite-horizon criterion; unbounded transition rates; occupation measure; history-dependent policies.

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1 Introduction

Continuous-time Markov decision processes (CTMDPs) have been applied in many areas, such as queueing systems, epidemiology, and telecommunication; see, for instance, [7, 17, 21]

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and the references therein. The optimality criteria for CTMDPs can be classified into the finite-horizon and infinite-horizon criteria. As we can see in the existing literature, the infinite-horizon criteria have been widely studied by many authors; see, for instance, [7–11, 19, 21] and their extensive references. Comparing with the infinite-horizon criteria, there exist few works on the finite-horizon criteria for CTMDPs, whose treatment is more difficult than that of infinite-horizon criteria. On the other hand, as we know, the finite-horizon criteria for discrete-time MDPs have found rich applications to portfolio investment, inventory management, highway pavement maintenance, etc.; see, for instance, [2, 4, 21]. In view of applications, sometimes it is more suitable to formulate the optimization models with finite-horizon criteria than those with infinite-horizon criteria. We study the unconstrained and constrained cases for CTMDPs under the finite-horizon expected total cost criterion in this paper. Our main goals are as follows:

- (i) Give conditions for the existence of optimal policies in the class of all randomized history-dependent policies for the unconstrained and constrained cases;
- (ii) Formulate the constrained optimization problem as a linear program;
- (iii) Establish the strong duality between the primal linear program and its dual program.

For the unconstrained case, we briefly describe the previous literature on the finite-horizon criteria for CTMDPs. The existence of a solution to the optimality equation is established in [18] for finite states and finite actions, in [2] for bounded transition rates and denumerable states, in [6, 20, 24] for bounded transition rates and Borel state spaces, in [3] for unbounded transition rates and denumerable states, and in [23] for unbounded transition rates and Borel spaces. It should be noted that all the aforementioned works restrict the discussions of the finite-horizon optimization problems to the class of all Markov policies. However, the decision-makers may make decisions basing on the past information. To consider the past information, the definition of a randomized history-dependent policy has been introduced in [8–11, 17, 19] to study the infinite-horizon criteria for CTMDPs.

In this paper we discuss the finite-horizon criteria with the randomized history-dependent policies. The state space is denumerable and the action space is a Polish space. The transition and cost rates are allowed to be unbounded from above and from below. The dynamic programming approach is used to prove the existence of optimal policies under the suitable conditions. We first give a new estimation of the weight function in the form of conditional expectation induced by the randomized history-dependent policies, which extends the results in [7–11, 19] (see Theorem 3.1). It should be mentioned that the extension of this estimation is nontrivial. Then we derive the analogue of the forward Kolmogorov equation in the form of conditional expectation by a technique of the dual predictable projection (see Theorem 3.2). Finally, applying the analogue of the forward Kolmogorov equation, we show that the

finite-horizon optimal value function is the unique solution to the optimality equation and obtain the existence of an optimal deterministic Markov policy, which extend the results in [2, 3, 6, 18, 20, 23, 24] from the class of all Markov policies to the more general class of all randomized history-dependent policies (see Theorem 4.2). It is worthy to point out that since the controlled state process does not have the Markov property under any randomized history-dependent policy and the finite-horizon optimal value function includes a time variable, the analyses are more difficult and complicated than those of the infinite-horizon criteria with the randomized history-dependent policies and the finite-horizon criteria with the Markov policies. Moreover, the fixed point theorem and uniformization techniques are inapplicable to the case of unbounded transition rates.

For the constrained case, the optimality criterion to be minimized is the finite-horizon expected total costs, and the constraints are imposed on the similar finite-horizon expected total costs. We employ the convex analytic approach by introducing the occupation measures of the finite-horizon criteria. Under suitable conditions, we give an equivalent characterization of the occupation measures and show that the set of all occupation measures is convex, compact, and metrizable in the w -weak topology (see Theorem 5.1). From this equivalent characterization, we conclude that for each occupation measure generated by a randomized history-dependent policy, there exists an occupation measure generated by a randomized Markov policy equal to it. Moreover, the constrained optimization problem can be reformulated as a linear program. Applying the Weierstrass theorem, we obtain the existence of a constrained-optimal randomized Markov policy (see Theorem 5.2). Finally, we develop the dual program of the linear program, and establish the strong duality between the primal linear program and its dual program (see Theorem 5.3).

The rest of this paper is organized as follows. In Section 2, we introduce the control model and optimization problem. In Section 3, we give optimality conditions for the existence of optimal policies and some preliminary results. The main results for the unconstrained and constrained optimization problems are presented in Sections 4 and 5, respectively. In Section 6, we illustrate our main results with a controlled birth and death system.

2 The control model and optimization problem

The primitive data of the control model in this paper are as follows:

$$\{S, A, (A(i), i \in S), q(j|i, a), c_0(i, a)(c_n(i, a), d_n, 1 \leq n \leq N)\},$$

where the state space S is assumed to be a denumerable set endowed with discrete topology and the action space A is assumed to be a Polish space with Borel σ -algebra $\mathcal{B}(A)$. $A(i) \in \mathcal{B}(A)$ denotes the set of admissible actions when the state of the system is $i \in S$. Define

$K := \{(i, a) \mid i \in S, a \in A(i)\}$ which contains all the feasible state-action pairs. The transition rates $q(j|i, a)$ are supposed to satisfy the following properties:

- For each fixed $i, j \in S$, $q(j|i, a)$ is measurable in $a \in A(i)$;
- $q(j|i, a) \geq 0$ for all $(i, a) \in K$ and $j \neq i$;
- $\sum_{j \in S} q(j|i, a) = 0$ for all $(i, a) \in K$;
- $q^*(i) := \sup_{a \in A(i)} |q(i|i, a)| < \infty$ for all $i \in S$.

Finally, the real-valued cost functions $c_n(i, a)$ ($0 \leq n \leq N$) on K are assumed to be measurable in $a \in A(i)$ for each $i \in S$ and the real numbers d_n ($1 \leq n \leq N$) denote the constraints imposed on the finite-horizon expected total costs.

Let $S_\infty := S \cup \{i_\infty\}$ with an isolated point $i_\infty \notin S$, $\mathbb{R}_+ := (0, +\infty)$, $\mathbb{R}_+^0 := [0, +\infty)$, $\Omega^0 := (S \times \mathbb{R}_+)^{\infty}$, and $\Omega := \Omega^0 \cup \{(i_0, \theta_1, i_1, \dots, \theta_{m-1}, i_{m-1}, \infty, i_\infty, \infty, i_\infty, \dots) \mid i_0 \in S, i_l \in S, \theta_l \in \mathbb{R}_+ \text{ for each } 1 \leq l \leq m-1, m \geq 2\}$. Hence, we obtain a measurable space (Ω, \mathcal{F}) in which \mathcal{F} is the standard Borel σ -algebra. Define the maps on (Ω, \mathcal{F}) below: for each $\omega = (i_0, \theta_1, i_1, \dots) \in \Omega$, let $T_0(\omega) := 0$, $X_0(\omega) := i_0$; for $m \geq 1$, let $\Theta_m(\omega) := \theta_m$, $X_m(\omega) := i_m$, $T_m(\omega) := \theta_1 + \theta_2 + \dots + \theta_m$, $T_\infty(\omega) := \lim_{m \rightarrow \infty} T_m(\omega)$, and

$$\xi_t(\omega) := \sum_{m \geq 0} I_{\{T_m \leq t < T_{m+1}\}} i_m + I_{\{T_\infty \leq t\}} i_\infty \quad \text{for all } t \geq 0,$$

where I_D denotes the indicator function of a set D . $\{T_m\}_{m \geq 0}$ are the jump epochs, and X_m is the state of the process $\{\xi_t, t \geq 0\}$ on $[T_m, T_{m+1})$. Since we do not intend to consider the process after T_∞ , it is regarded to be absorbed in the state i_∞ . Hence, we write $q(i_\infty|i_\infty, a_\infty) = 0$, where a_∞ is an isolated point. Moreover, we set $A_\infty := A \cup \{a_\infty\}$, $A(i_\infty) := \{a_\infty\}$, $\mathcal{F}_t := \sigma(\{T_m \leq s, X_m = j\} : j \in S, s \leq t, m \geq 0)$ for all $t \geq 0$, $\mathcal{F}_{s-} := \bigvee_{0 \leq t < s} \mathcal{F}_t$ (i.e., the smallest σ -algebra containing all the σ -algebras $\{\mathcal{F}_t, 0 \leq t < s\}$), and $\mathcal{P} := \sigma(B \times \{0\} (B \in \mathcal{F}_0), B \times (s, \infty) (B \in \mathcal{F}_{s-}, s > 0))$ which is the σ -algebra of predictable sets on $\Omega \times \mathbb{R}_+^0$ with respect to $\{\mathcal{F}_t\}_{t \geq 0}$.

To define the optimality criteria, we introduce the definition of a policy below.

Definition 2.1. A \mathcal{P} -measurable transition probability $\pi(\cdot \mid \omega, t)$ on $(A_\infty, \mathcal{B}(A_\infty))$, concentrated on $A(\xi_{t-}(\omega))$, is called a randomized history-dependent policy. A policy is called randomized Markov if there exists a kernel φ on A_∞ given $S_\infty \times \mathbb{R}_+^0$ such that $\pi(\cdot \mid \omega, t) = \varphi(\cdot \mid \xi_{t-}(\omega), t)$. A policy is called deterministic Markov if there exists a measurable function f on $S_\infty \times \mathbb{R}_+^0$ with $f(i, t) \in A(i)$ for all $(i, t) \in S_\infty \times \mathbb{R}_+^0$, such that $\pi(\cdot \mid \omega, t) = \delta_{f(\xi_{t-}(\omega), t)}(\cdot)$, where $\delta_x(\cdot)$ is a Dirac measure concentrated at x .

We denote by Π the set of all randomized history-dependent policies, by Π^M the set of all randomized Markov policies, and by Π^D the set of all deterministic Markov policies.

For any $\pi \in \Pi$, we define the random measure

$$\nu^\pi(\omega, dt, j) := \left[\int_A \pi(da|\omega, t) q(j|\xi_{t-}(\omega), a) I_{\{j \neq \xi_{t-}(\omega)\}} \right] dt \quad (2.1)$$

for any $j \in S$. Then, we have that this random measure is predictable, and $\nu^\pi(\omega, \{t\} \times S) = \nu^\pi(\omega, [T_\infty, \infty) \times S) = 0$. Hence, for any $\pi \in \Pi$ and any initial distribution γ on S , by Theorem 4.27 in [17], there exists a unique probability measure P_γ^π on (Ω, \mathcal{F}) such that $P_\gamma^\pi(\xi_0 = i) = \gamma(i)$, and with respect to P_γ^π , ν^π is the dual predictable projection of random measure on $\mathbb{R}_+ \times S$

$$\mu(\omega, dt, i) = \sum_{m \geq 1} I_{\{T_m < \infty\}} I_{\{X_m = i\}} \delta_{T_m}(dt). \quad (2.2)$$

Therefore, we obtain a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P_\gamma^\pi)$, which is always assumed to be complete. When $\gamma(j) = \delta_i(j)$ for all $j \in S$, we write P_γ^π as P_i^π . The expectation operators with respect to P_γ^π and P_i^π are denoted as E_γ^π and E_i^π , respectively.

For each initial state $i \in S$, $\pi \in \Pi$, and any fixed initial distribution γ on S , we define the finite-horizon expected total costs from time 0 to the fixed terminal time $T > 0$ as follows:

$$V_n(i, \pi) := E_i^\pi \left[\int_0^T \int_A c_n(\xi_{t-}, a) \pi(da|\omega, t) dt \right],$$

$$V_n(\pi) := E_\gamma^\pi \left[\int_0^T \int_A c_n(\xi_{t-}, a) \pi(da|\omega, t) dt \right]$$

for all $n = 0, 1, \dots, N$, provided that the integrals are well defined.

A policy $\pi^* \in \Pi$ is said to be finite-horizon optimal if $V_0(i, \pi^*) = \inf_{\pi \in \Pi} V_0(i, \pi)$ for all $i \in S$.

Now we state the constrained optimization problem considered in this paper below:

$$\text{Minimize } V_0(\pi) \text{ over } U := \{\pi \in \Pi \mid V_n(\pi) \leq d_n, \ 1 \leq n \leq N\}. \quad (2.3)$$

Definition 2.2. A policy $\pi^* \in U$ is said to be constrained-optimal if $V_0(\pi^*) = \inf_{\pi \in U} V_0(\pi)$.

3 Preliminaries

In this section, we will give optimality conditions for the existence of optimal policies and some preliminary results to prove our main results.

To avoid the explosiveness of the process $\{\xi_t, t \geq 0\}$, we need the following condition from [7, 19].

Assumption 3.1. *There exist a weight function $w \geq 1$ on S , and constants $\rho_1 > 0$, $b_1 \geq 0$, and $L > 0$, such that*

- (i) $\sum_{j \in S} w(j)q(j|i, a) \leq \rho_1 w(i) + b_1$ for all $(i, a) \in K$.
- (ii) $q^*(i) \leq Lw(i)$ for all $i \in S$.

In order to guarantee the finiteness of finite-horizon expected total cost criteria, we also consider the conditions below, which are widely used in [7–11, 19].

Assumption 3.2. (i) $\gamma(w) := \sum_{i \in S} w(i)\gamma(i) < \infty$, where w comes from Assumption 3.1.
(ii) There exists a constant $M > 0$ such that $|c_n(i, a)| \leq Mw(i)$ for all $(i, a) \in K$ and $n = 0, 1, \dots, N$.

Under the above two assumptions, we have the following statements.

Lemma 3.1. Suppose that Assumptions 3.1 and 3.2 hold. Then for each $\pi \in \Pi$, we have

- (a) $P_\gamma^\pi(T_\infty = \infty) = 1$ and $P_\gamma^\pi(\xi_t \in S) = 1$ for all $t \geq 0$.
- (b) $E_i^\pi[w(\xi_t)] \leq e^{\rho_1 t} w(i) + \frac{b_1}{\rho_1}(e^{\rho_1 t} - 1)$ for all $i \in S$ and $t \geq 0$.
- (c) $|V_n(i, \pi)| \leq MT \left[e^{\rho_1 T} w(i) + \frac{b_1}{\rho_1}(e^{\rho_1 T} - 1) \right]$ for all $i \in S$ and $n = 0, 1, \dots, N$.
- (d) $|V_0(\pi)| \leq MT \left[e^{\rho_1 T} \gamma(w) + \frac{b_1}{\rho_1}(e^{\rho_1 T} - 1) \right]$ for all $n = 0, 1, \dots, N$.

Proof. Parts (a) and (b) follow from Proposition 2.1 in [19].

(c) By Assumption 3.2(ii), we obtain

$$\begin{aligned} |V_n(i, \pi)| &\leq E_i^\pi \left[\int_0^T \int_A |c_n(\xi_{t-}, a)| \pi(da|\omega, t) dt \right] \\ &\leq ME_i^\pi \left[\int_0^T w(\xi_t) dt \right] \leq MT \left[e^{\rho_1 T} w(i) + \frac{b_1}{\rho_1}(e^{\rho_1 T} - 1) \right] \end{aligned}$$

for all $i \in S$, $\pi \in \Pi$, and $n = 0, 1, \dots, N$, where the last inequality follows from part (b).

(d) Part (d) follows immediately from part (c). \square

In addition to Assumptions 3.1 and 3.2, we also need the following conditions to ensure the existence of optimal policies.

Assumption 3.3. (i) There exist constants $\rho_2 > 0$, $\rho_3 > 0$, $b_2 \geq 0$, and $b_3 \geq 0$ such that

$$\sum_{i \in S} w^2(j)q(j|i, a) \leq \rho_2 w^2(i) + b_2, \quad \text{and} \quad \sum_{j \in S} w^3(j)q(j|i, a) \leq \rho_3 w^3(i) + b_3$$

for all $(i, a) \in K$, where w comes from Assumption 3.1.

(ii) For each $i \in S$, the set $A(i)$ is compact.

(iii) For each fixed $i, j \in S$ and $n = 0, 1, \dots, N$, the functions $c_n(i, a)$, $q(j|i, a)$ and $\sum_{k \in S} w(k)q(k|i, a)$ are continuous in $a \in A(i)$.

Finally, we give the following assertions which are used to prove our main results.

Lemma 3.2. Fix any $i \in S$, $\pi \in \Pi$, and $s \in \mathbb{R}_+$. Let $\mathcal{F}_{T_n} := \sigma(X_m, T_m, 0 \leq m \leq n)$. For any integrable random variable Z on $(\Omega, \mathcal{F}, P_i^\pi)$, we have

$$E_i^\pi[Z | \mathcal{F}_{T_n}, s < T_{n+1}] I_{\{s \in [T_n, T_{n+1})\}} = \frac{E_i^\pi[Z I_{\{s < T_{n+1}\}} | \mathcal{F}_{T_n}]}{E_i^\pi[I_{\{s < T_{n+1}\}} | \mathcal{F}_{T_n}]} I_{\{s \in [T_n, T_{n+1})\}},$$

where we make the convention that $\frac{0}{0} = 0$.

Proof. Since $\{s \in [T_n, T_{n+1})\} \in \sigma(\mathcal{F}_{T_n}, \{s \in [T_n, T_{n+1})\})$, we have

$$E_i^\pi[Z | \mathcal{F}_{T_n}, s < T_{n+1}] I_{\{s \in [T_n, T_{n+1})\}} = E_i^\pi[Z I_{\{s \in [T_n, T_{n+1})\}} | \mathcal{F}_{T_n}, s < T_{n+1}].$$

For any $B \in \mathcal{F}_{T_n}$, straightforward calculations yield

$$\begin{aligned} & \int_{B \cap \{s < T_{n+1}\}} \frac{E_i^\pi[Z I_{\{s < T_{n+1}\}} | \mathcal{F}_{T_n}]}{E_i^\pi[I_{\{s < T_{n+1}\}} | \mathcal{F}_{T_n}]} I_{\{s \in [T_n, T_{n+1})\}} dP_i^\pi \\ &= E_i^\pi \left[I_{\{s < T_{n+1}\}} \frac{E_i^\pi[Z I_{B \cap \{s < T_{n+1}\} \cap \{T_n \leq s\}} | \mathcal{F}_{T_n}]}{E_i^\pi[I_{\{s < T_{n+1}\}} | \mathcal{F}_{T_n}]} \right] \\ &= E_i^\pi \left[E_i^\pi \left[I_{\{s < T_{n+1}\}} \frac{E_i^\pi[Z I_{B \cap \{s < T_{n+1}\} \cap \{T_n \leq s\}} | \mathcal{F}_{T_n}]}{E_i^\pi[I_{\{s < T_{n+1}\}} | \mathcal{F}_{T_n}]} \middle| \mathcal{F}_{T_n} \right] \right] \\ &= E_i^\pi \left[E_i^\pi[I_{\{s < T_{n+1}\}} | \mathcal{F}_{T_n}] \frac{E_i^\pi[Z I_{B \cap \{s < T_{n+1}\} \cap \{T_n \leq s\}} | \mathcal{F}_{T_n}]}{E_i^\pi[I_{\{s < T_{n+1}\}} | \mathcal{F}_{T_n}]} \right] \\ &= E_i^\pi[Z I_{B \cap \{s < T_{n+1}\} \cap \{T_n \leq s\}}] \\ &= \int_{B \cap \{s < T_{n+1}\}} Z I_{\{s \in [T_n, T_{n+1})\}} dP_i^\pi, \end{aligned}$$

where the first equality is due to the fact that $B \in \mathcal{F}_{T_n}$ and $\{T_n \leq s\} \in \mathcal{F}_{T_n}$, and

$$\int_{B \cap \{T_{n+1} \leq s\}} \frac{E_i^\pi[Z I_{\{s < T_{n+1}\}} | \mathcal{F}_{T_n}]}{E_i^\pi[I_{\{s < T_{n+1}\}} | \mathcal{F}_{T_n}]} I_{\{s \in [T_n, T_{n+1})\}} dP_i^\pi = \int_{B \cap \{T_{n+1} \leq s\}} Z I_{\{s \in [T_n, T_{n+1})\}} dP_i^\pi = 0.$$

By the monotone class theorem, we have that for any $C \in \sigma(\mathcal{F}_{T_n}, \{s \in [T_n, T_{n+1})\})$,

$$\int_C \frac{E_i^\pi[Z I_{\{s < T_{n+1}\}} | \mathcal{F}_{T_n}]}{E_i^\pi[I_{\{s < T_{n+1}\}} | \mathcal{F}_{T_n}]} I_{\{s \in [T_n, T_{n+1})\}} dP_i^\pi = \int_C Z I_{\{s \in [T_n, T_{n+1})\}} dP_i^\pi.$$

Hence, the assertion follows from the definition of conditional expectation. \square

Employing Lemma 3.2, we have the following result.

Theorem 3.1. Suppose that Assumption 3.1 holds. Then for any $i \in S$, $\pi \in \Pi$, and $s < t$, the following statements hold.

$$(a) \quad E_i^\pi[w(\xi_t)I_{\{t < T_{m+1}\}} | \mathcal{F}_s] \leq (e^{\rho_1(t-s)} + 1)w(\xi_s) + \frac{b_1}{\rho_1}(e^{\rho_1(t-s)} - 1) \text{ for all } m = 0, 1, \dots$$

$$(b) \quad E_i^\pi[w(\xi_t) | \xi_s] \leq (e^{\rho_1(t-s)} + 1)w(\xi_s) + \frac{b_1}{\rho_1}(e^{\rho_1(t-s)} - 1).$$

Proof. (a) Because $s < t$, and the sets $\{T_n \leq s\}$, $\{T_\infty \leq s\}$ are in \mathcal{F}_s , we have

$$\begin{aligned} & E_i^\pi[w(\xi_t)I_{\{t < T_{m+1}\}} | \mathcal{F}_s] \\ &= \sum_{n=0}^{\infty} E_i^\pi[w(\xi_t)I_{\{t < T_{m+1}\}} | \mathcal{F}_s]I_{\{s \in [T_n, T_{n+1})\}} + E_i^\pi[w(\xi_t)I_{\{t < T_{m+1}\}} | \mathcal{F}_s]I_{\{T_\infty \leq s\}} \\ &= \sum_{n=0}^m E_i^\pi[w(\xi_t)I_{\{t < T_{m+1}\}} | \mathcal{F}_s]I_{\{s \in [T_n, T_{n+1})\}}. \end{aligned} \quad (3.1)$$

Below we fix $n \in \{0, 1, \dots, m\}$, and set $\mathcal{F}_{T_n} := \sigma(X_l, T_l, 0 \leq l \leq n)$. Since $\{s \in [T_n, T_{n+1})\} \in \mathcal{F}_s \cap \sigma(\mathcal{F}_{T_n}, \{s \in [T_n, T_{n+1})\})$ and $\{s \in [T_n, T_{n+1})\} \cap \mathcal{F}_s = \{s \in [T_n, T_{n+1})\} \cap \sigma(\mathcal{F}_{T_n}, \{s \in [T_n, T_{n+1})\})$, by Lemma 6.2 in [15], we obtain

$$\begin{aligned} E_i^\pi[w(\xi_t)I_{\{t < T_{m+1}\}} | \mathcal{F}_s]I_{\{s \in [T_n, T_{n+1})\}} &= E_i^\pi[w(\xi_t)I_{\{t < T_{m+1}\}} | \mathcal{F}_{T_n}, s \in [T_n, T_{n+1})]I_{\{s \in [T_n, T_{n+1})\}} \\ &= E_i^\pi[w(\xi_t)I_{\{t < T_{m+1}\}} | \mathcal{F}_{T_n}, s < T_{n+1}]I_{\{s \in [T_n, T_{n+1})\}} \\ &= \frac{E_i^\pi[w(\xi_t)I_{\{t < T_{m+1}\} \cap \{s < T_{n+1}\}} | \mathcal{F}_{T_n}]}{E_i^\pi[I_{\{s < T_{n+1}\}} | \mathcal{F}_{T_n}]} I_{\{s \in [T_n, T_{n+1})\}}, \end{aligned} \quad (3.2)$$

where the second equality holds because $\{T_n \leq s\} \in \mathcal{F}_{T_n}$, and the third one follows from Lemma 3.2. Moreover, direct computations yield

$$\begin{aligned} & E_i^\pi[w(\xi_t)I_{\{t < T_{m+1}\} \cap \{s < T_{n+1}\}} | \mathcal{F}_{T_n}]I_{\{s \in [T_n, T_{n+1})\}} \\ &= E_i^\pi[E_i^\pi[w(\xi_t)I_{\{t < T_{m+1}\} \cap \{s < T_{n+1}\}} | \mathcal{F}_{T_{n+1}}] | \mathcal{F}_{T_n}]I_{\{s \in [T_n, T_{n+1})\}} \\ &= E_i^\pi[E_i^\pi[w(\xi_t)I_{\{t < T_{m+1}\}} | \mathcal{F}_{T_{n+1}}]I_{\{T_n \leq s < T_{n+1}\}} | \mathcal{F}_{T_n}]I_{\{s \in [T_n, T_{n+1})\}} \\ &\leq E_i^\pi \left[\left\{ I_{\{T_{n+1} \leq t\}} h(T_{n+1}, X_{n+1}, t) + \sum_{l=1}^{n+1} I_{\{T_{l-1} \leq t < T_l\}} w(X_{l-1}) \right\} I_{\{T_n \leq s < T_{n+1}\}} \middle| \mathcal{F}_{T_n} \right] I_{\{s \in [T_n, T_{n+1})\}} \\ &= E_i^\pi [I_{\{T_{n+1} \leq t\}} I_{\{T_n \leq s < T_{n+1}\}} h(T_{n+1}, X_{n+1}, t) + I_{\{T_n \leq t < T_{n+1}\}} I_{\{T_n \leq s < T_{n+1}\}} w(X_n) | \mathcal{F}_{T_n}] I_{\{s \in [T_n, T_{n+1})\}} \\ &= E_i^\pi [I_{\{s < T_{n+1} \leq t\}} h(T_{n+1}, X_{n+1}, t) + I_{\{t < T_{n+1}\}} w(X_n) | \mathcal{F}_{T_n}] I_{\{s \in [T_n, T_{n+1})\}}, \end{aligned}$$

where $h(\underline{t}, i, \bar{t}) := e^{\rho_1(\bar{t}-\underline{t})}w(i) + \frac{b_1}{\rho_1}(e^{\rho_1(\bar{t}-\underline{t})} - 1)$ for all $i \in S$, $0 \leq \underline{t} \leq \bar{t}$, and the inequality follows from equality (36) in [8]. Note that $I_{\{t < T_{n+1}\}} \leq I_{\{s < T_{n+1}\}}$. On one hand, we get

$$\frac{E_i^\pi [I_{\{t < T_{n+1}\}} w(X_n) | \mathcal{F}_{T_n}] I_{\{s \in [T_n, T_{n+1})\}}}{E_i^\pi [I_{\{s < T_{n+1}\}} | \mathcal{F}_{T_n}]} = \frac{E_i^\pi [I_{\{t < T_{n+1}\}} | \mathcal{F}_{T_n}] I_{\{s \in [T_n, T_{n+1})\}}}{E_i^\pi [I_{\{s < T_{n+1}\}} | \mathcal{F}_{T_n}]} w(X_n)$$

$$\leq w(X_n)I_{\{s \in [T_n, T_{n+1}]\}}.$$

On the other hand, it follows from Lemma 3.3 in [14] that the function

$$\Lambda^\pi(j|\omega, t) := \int_A \pi(da|\omega, t) q(j|\xi_{t-}(\omega), a) I_{\{j \neq \xi_{t-}(\omega)\}}$$

has the representation below:

$$\Lambda^\pi(j|\omega, t) = I_{\{t=0\}} \Lambda^0(j|i_0) + \sum_{l \geq 0} I_{\{T_l(\omega) < t \leq T_{l+1}(\omega)\}} \Lambda^l(j|i_0, \theta_1, i_1, \dots, \theta_l, i_l, t - T_l(\omega)),$$

where $\Lambda^l(j|i_0, \theta_1, i_1, \dots, \theta_l, i_l, \tilde{t})$ are some nonnegative nonrandom measurable functions. Employing the construction of the measure P_i^π (see [8] for details), we have

$$\begin{aligned} & \frac{E_i^\pi [I_{\{s < T_{n+1} \leq t\}} h(T_{n+1}, X_{n+1}, t) | \mathcal{F}_{T_n}] I_{\{s \in [T_n, T_{n+1}]\}}}{E_i^\pi [I_{\{s < T_{n+1}\}} | \mathcal{F}_{T_n}]} \\ &= \int_{s-T_n}^{t-T_n} \left\{ e^{-\int_0^u \sum_{j \neq X_n} \Lambda^n(j|X_0, \Theta_1, X_1, \dots, \Theta_n, X_n, v) dv} \right. \\ & \quad \times \sum_{k \neq X_n} h(T_n + u, k, t) \Lambda^n(k|X_0, \Theta_1, X_1, \dots, \Theta_n, X_n, u) \Big\} du \\ & \quad \times \left(e^{-\int_0^{s-T_n} \sum_{j \neq X_n} \Lambda^n(j|X_0, \Theta_1, X_1, \dots, \Theta_n, X_n, v) dv} \right)^{-1} I_{\{s \in [T_n, T_{n+1}]\}} \\ &= \int_{s-T_n}^{t-T_n} \left\{ e^{-\int_{s-T_n}^u \sum_{j \neq X_n} \Lambda^n(j|X_0, \Theta_1, X_1, \dots, \Theta_n, X_n, v) dv} \right. \\ & \quad \times \sum_{k \neq X_n} h(T_n + u, k, t) \Lambda^n(k|X_0, \Theta_1, X_1, \dots, \Theta_n, X_n, u) \Big\} du I_{\{s \in [T_n, T_{n+1}]\}} \\ &\leq h(s - T_n, X_n, t - T_n) I_{\{s \in [T_n, T_{n+1}]\}} = h(0, X_n, t - s) I_{\{s \in [T_n, T_{n+1}]\}}, \end{aligned}$$

where the inequality follows from Lemma A.1 in [8]. Hence, we obtain

$$\frac{E_i^\pi [w(\xi_t) I_{\{t < T_{m+1}\} \cap \{s < T_{n+1}\}} | \mathcal{F}_{T_n}] I_{\{s \in [T_n, T_{n+1}]\}}}{E_i^\pi [I_{\{s < T_{n+1}\}} | \mathcal{F}_{T_n}]} \leq [h(0, X_n, t - s) + w(X_n)] I_{\{s \in [T_n, T_{n+1}]\}}. \quad (3.3)$$

Therefore, by (3.1)-(3.3), we have

$$\begin{aligned} E_i^\pi [w(\xi_t) I_{\{t < T_{m+1}\}} | \mathcal{F}_s] &\leq \sum_{n=0}^m [h(0, X_n, t - s) + w(X_n)] I_{\{s \in [T_n, T_{n+1}]\}} \\ &= \sum_{n=0}^m [h(0, \xi_s, t - s) + w(\xi_s)] I_{\{s \in [T_n, T_{n+1}]\}} \\ &\leq h(0, \xi_s, t - s) + w(\xi_s). \end{aligned}$$

(b) By part (a), we have

$$E_i^\pi [w(\xi_t) I_{\{t < T_{m+1}\}} | \xi_s] \leq (e^{\rho_1(t-s)} + 1)w(\xi_s) + \frac{b_1}{\rho_1}(e^{\rho_1(t-s)} - 1) \quad (3.4)$$

for all $m = 0, 1, 2, \dots$. Moreover, it follows from Proposition 2.1 in [19] that

$$P_i^\pi \left(\lim_{m \rightarrow \infty} w(\xi_t) I_{\{t < T_{m+1}\}} = w(\xi_t) \right) = 1.$$

Hence, the assertion follows from (3.4), Lemma 3.1(a) and the dominated convergence theorem of conditional expectation. This completes the proof of the theorem. \square

Remark 3.1. Theorem 3.1 presents a *new* estimation on the so-called weight function w in the form of conditional expectation induced by the randomized history-dependent policies, which generalizes the estimation in the case of conditional expectation induced by the Markov policies in [7] and the case of expectation induced by the randomized history-dependent policies in [8–11, 19]. Moreover, since the state process does not have the Markov property under any randomized history-dependent policy, the technique in [7] is inapplicable here.

The following assertion extends the analogue of the forward Kolmogorov equation in [8–11, 19] to that in the form of conditional expectation.

Theorem 3.2. *Suppose that Assumption 3.1 holds. Then for any $i \in S$, $B \subseteq S$, $\pi \in \Pi$, and $s < t$, we have*

$$P_i^\pi(\xi_t \in B | \mathcal{F}_s) = I_{\{\xi_s \in B\}} + E_i^\pi \left[\int_s^t \int_A q(B | \xi_u, a) \pi(da | \omega, u) du \middle| \mathcal{F}_s \right].$$

Proof. Fix any $i \in S$, $B \subseteq S$, $\pi \in \Pi$, and $s \in \mathbb{R}_+^0$. Let $\nu_1^\pi(\omega, dt, j) := I_{(s, \infty)}(t) \nu^\pi(\omega, dt, j)$ and $\mu_1(\omega, dt, j) := I_{(s, \infty)}(t) \mu(\omega, dt, j)$, where the random measures ν^π and μ are as in (2.1) and (2.2), respectively. Since $Y(\omega, t) := I_{(s, \infty)}(t)$ is a predictable process, the dual predictable projection of μ_1 is ν_1^π . Define the random measures below:

$$\tilde{\mu}_1(\omega, dt, j) := \sum_{m \geq 1} I_{\{T_m < \infty\}} I_{\{X_{m-1} = j\}} I_{(s, \infty)}(t) \delta_{T_m}(dt),$$

and

$$\tilde{\nu}_1^\pi(\omega, dt, j) := I_{(s, \infty)}(t) \int_A (-q(\xi_{t-} | \xi_{t-}, a)) \pi(da | \omega, t) \delta_{\xi_{t-}}(j) dt.$$

Then, by Lemma 4 in [16], we have that $\tilde{\nu}_1^\pi$ is the dual predictable projection of $\tilde{\mu}_1$ with respect to P_i^π . From the proof of Theorem 3.1 in [8], we have

$$E_i^\pi[\mu_1((0, t], B)] < \infty \text{ and } E_i^\pi[\tilde{\mu}_1((0, t], B)] < \infty \text{ for all } t > s.$$

Obviously, we have the following equation

$$I_{\{\xi_t \in B\}} = I_{\{\xi_s \in B\}} + \mu_1((0, t], B) - \tilde{\mu}_1((0, t], B).$$

Hence, taking conditional expectation in the both sides of the last equation, we obtain

$$E_i^\pi[I_{\{\xi_t \in B\}} | \mathcal{F}_s] = I_{\{\xi_s \in B\}} + E_i^\pi[\mu_1((0, t], B) | \mathcal{F}_s] - E_i^\pi[\tilde{\mu}_1((0, t], B) | \mathcal{F}_s]$$

$$\begin{aligned}
&= I_{\{\xi_s \in B\}} + E_i^\pi [\nu_1^\pi((0, t], B) | \mathcal{F}_s] - E_i^\pi [\tilde{\nu}_1^\pi((0, t], B) | \mathcal{F}_s] \\
&= I_{\{\xi_s \in B\}} + E_i^\pi \left[\int_s^t \int_A q(B \setminus \{\xi_u\} | \xi_u, a) \pi(da | \omega, u) du \middle| \mathcal{F}_s \right] \\
&\quad + E_i^\pi \left[\int_s^t \int_A q(\xi_u | \xi_u, a) \pi(da | \omega, u) I_{\{\xi_u \in B\}} du \middle| \mathcal{F}_s \right],
\end{aligned}$$

where the second equality is due to the fact that $\mu_1((0, t], B) - \nu_1^\pi((0, t], B)$ and $\tilde{\mu}_1((0, t], B) - \tilde{\nu}_1^\pi((0, t], B)$ are martingales. \square

4 Dynamic programming for the unconstrained case

In this section, we will use the dynamic programming approach to show the existence of optimal policies. To this end, we introduce the following notation.

For any $s \in [0, T]$, a function g defined on $S \times [s, T]$ is said to be $[s, T]$ -uniformly w^2 -bounded if it is measurable and there exists a constant $\tilde{M} > 0$ such that $|g(i, t)| \leq \tilde{M}w^2(i)$ for all $(i, t) \in S \times [s, T]$.

For any $i, j \in S$, $\pi \in \Pi$, and $s \in [0, T]$, define the set

$$\begin{aligned}
\mathcal{H}_{s,j}^{i,\pi} &:= \left\{ g : g \text{ is } [s, T]\text{-uniformly } w^2\text{-bounded and satisfies} \right. \\
&\quad E_i^\pi \left[\int_s^T \int_A \sum_{k \in S} \left[\int_t^T g(k, v) dv \right] q(k | \xi_t, a) \pi(da | \omega, t) dt \middle| \xi_s = j \right] \\
&\quad \left. = E_i^\pi \left[\int_s^T g(\xi_t, t) dt \middle| \xi_s = j \right] - \int_s^T g(j, t) dt \right\}. \tag{4.1}
\end{aligned}$$

For each initial state $i \in S$, $\pi \in \Pi$, and $s \in [0, T]$, the expected total cost from $s \geq 0$ to the terminal time $T > 0$ and the corresponding optimal value function are defined as

$$U(i, j, s, \pi) := E_i^\pi \left[\int_s^T \int_A c_0(\xi_{t-}, a) \pi(da | \omega, t) dt \middle| \xi_s = j \right] \quad \text{and} \quad U^*(i, j, s) := \inf_{\pi \in \Pi} U(i, j, s, \pi)$$

for all $j \in S$, respectively. In particular, when $s = 0$, we have $U(i, i, 0, \pi) = V_0(i, \pi)$.

Then we have the following *new* assertion on the property of $\mathcal{H}_{s,j}^{i,\pi}$.

Theorem 4.1. *Under Assumptions 3.1 and 3.3(i), the following statement holds: for any $i, j \in S$, $\pi \in \Pi$ and $s \in [0, T]$, the set $\mathcal{H}_{s,j}^{i,\pi}$ in (4.1) contains all $[s, T]$ -uniformly w^2 -bounded functions.*

Proof. Fix any $i, j \in S$, $\pi \in \Pi$, and $s \in [0, T]$. By Proposition 2.1 in [19], we obtain

$$E_i^\pi \left[\int_s^T \int_A \sum_{k \in S} w^2(k) |q(k | \xi_t, a)| \pi(da | \omega, t) dt \right]$$

$$\begin{aligned}
&\leq (\rho_2 + b_2 + 2L)E_i^\pi \left[\int_s^T w^3(\xi_t) dt \right] \\
&\leq (\rho_2 + b_2 + 2L) \int_s^T \left[e^{\rho_3 t} w^3(i) + \frac{b_3}{\rho_3} (e^{\rho_3 t} - 1) \right] dt \\
&\leq (\rho_2 + b_2 + 2L)T \left[e^{\rho_3 T} w^3(i) + \frac{b_3}{\rho_3} (e^{\rho_3 T} - 1) \right].
\end{aligned}$$

If g is $[s, T]$ -uniformly w^2 -bounded, from the last inequality, we have

$$E_i^\pi \left[\int_s^T \int_A \sum_{k \in S} \left| \int_t^T g(k, v) dv \right| q(k|\xi_t, a) \pi(da|\omega, t) dt \middle| \xi_s = j \right] < \infty.$$

Hence, $E_i^\pi \left[\int_s^T \int_A \sum_{k \in S} \left[\int_t^T g(k, v) dv \right] q(k|\xi_t, a) \pi(da|\omega, t) dt \middle| \xi_s = j \right]$ is well defined. Using the similar arguments, we see that $E_i^\pi \left[\int_s^T g(\xi_t, t) dt \middle| \xi_s = j \right]$ is well defined.

Let $\mathcal{C} := \{B \times [t_*, t^*] : B \subseteq S, s \leq t_* \leq t^* \leq T\}$. Then, it is obvious that \mathcal{C} is a π -system and $S \times [s, T] \in \mathcal{C}$. Below we will use the monotone class theorem to show that $\mathcal{H}_{s,j}^{i,\pi}$ contains all the bounded measurable functions on $S \times [s, T]$.

(i) For any $B \times [t_*, t^*] \in \mathcal{C}$, we will show that $I_B(k)I_{[t_*, t^*]}(t) \in \mathcal{H}_{s,j}^{i,\pi}$. Set $t_1 \vee t_2 := \max\{t_1, t_2\}$ and $t_1 \wedge t_2 := \min\{t_1, t_2\}$. Direct calculations yield

$$\begin{aligned}
&E_i^\pi \left[\int_s^T \int_A \sum_{k \in S} \left[\int_t^T I_B(k)I_{[t_*, t^*]}(v) dv \right] q(k|\xi_t, a) \pi(da|\omega, t) dt \middle| \xi_s = j \right] \\
&= E_i^\pi \left[\int_s^T [t^* - (t_* \vee t) \wedge t^*] \int_A q(B|\xi_t, a) \pi(da|\omega, t) dt \middle| \xi_s = j \right] \\
&= E_i^\pi \left[\int_s^{t_*} [t^* - (t_* \vee t) \wedge t^*] \int_A q(B|\xi_t, a) \pi(da|\omega, t) dt \middle| \xi_s = j \right] \\
&\quad + E_i^\pi \left[\int_{t_*}^{t^*} [t^* - (t_* \vee t) \wedge t^*] \int_A q(B|\xi_t, a) \pi(da|\omega, t) dt \middle| \xi_s = j \right] \\
&= (t^* - t_*) E_i^\pi \left[\int_s^{t_*} \int_A q(B|\xi_t, a) \pi(da|\omega, t) dt \middle| \xi_s = j \right] \\
&\quad + E_i^\pi \left[\int_{t_*}^{t^*} (t^* - t) \int_A q(B|\xi_t, a) \pi(da|\omega, t) dt \middle| \xi_s = j \right] \\
&= E_i^\pi \left[\int_{t_*}^{t^*} \int_s^t \int_A q(B|\xi_v, a) \pi(da|\omega, v) dv dt \middle| \xi_s = j \right] \\
&= \int_{t_*}^{t^*} E_i^\pi \left[\int_s^t \int_A q(B|\xi_v, a) \pi(da|\omega, v) dv \middle| \xi_s = j \right] dt \\
&= \int_{t_*}^{t^*} E_i^\pi [I_B(\xi_t) | \xi_s = j] dt - \int_{t_*}^{t^*} I_B(j) dt \\
&= E_i^\pi \left[\int_s^T I_B(\xi_t) I_{[t_*, t^*]}(t) dt \middle| \xi_s = j \right] - \int_s^T I_B(j) I_{[t_*, t^*]}(t) dt,
\end{aligned}$$

where the fourth equality is due to the integration by parts, and the sixth one follows from Theorem 3.2. Hence, we have $I_B(k)I_{[t_*, t^*]}(t) \in \mathcal{H}_{s,j}^{i,\pi}$.

(ii) If $0 \leq g_n \in \mathcal{H}_{s,j}^{i,\pi}$ ($n = 1, 2, \dots$), $g_n \uparrow g_0$ and g_0 is bounded, applying the monotone convergence theorem, we have $g_0 \in \mathcal{H}_{s,j}^{i,\pi}$.

Obviously, $\mathcal{H}_{s,j}^{i,\pi}$ is a linear space. Hence, (i), (ii) and the monotone class theorem give that $\mathcal{H}_{s,j}^{i,\pi}$ contains all measurable bounded functions on $S \times [s, T]$. Therefore, the desired assertion follows from the same technique of Lemma 3.7 in [23]. \square

Using Theorem 4.1, we obtain the main result on the existence of finite-horizon optimal policies for the case of unbounded transition and cost rates below.

Theorem 4.2. *Suppose that Assumptions 3.1, 3.2(ii), and 3.3 hold. Then we have*

- (a) *For any given $i \in S$, the function $U^*(i, \cdot)$ is the unique solution in $B_w(S \times [0, T])$ to the following equation: for each $j \in S$ and $s \in [0, T]$,*

$$g(j, s) = \int_s^T \inf_{a \in A(j)} \left\{ c_0(j, a) + \sum_{k \in S} g(k, t) q(k|j, a) \right\} dt, \quad (4.2)$$

where $B_w(S \times [0, T])$ denotes the set of all real-valued measurable functions g on $S \times [0, T]$ with $\sup_{j \in S} \sup_{s \in [0, T]} \frac{|g(j, s)|}{w(j)} < \infty$.

- (b) *There exists an optimal deterministic Markov policy π_T^* (depending on T).*

Proof. (a) Fix any $i \in S$, $\pi \in \Pi$, and $s \in [0, T]$. Then, direct calculations give

$$\begin{aligned} |U(i, j, s, \pi)| &\leq E_i^\pi \left[\int_s^T \int_A |c_0(\xi_{t-}, a)| \pi(da|\omega, t) dt \middle| \xi_s = j \right] \\ &\leq M E_i^\pi \left[\int_s^T w(\xi_t) dt \middle| \xi_s = j \right] \\ &= M \int_s^T E_i^\pi [w(\xi_t) | \xi_s = j] dt \\ &\leq M \int_s^T \left[(e^{\rho_1(t-s)} + 1)w(j) + \frac{b_1}{\rho_1}(e^{\rho_1(t-s)} - 1) \right] dt \\ &\leq MT \left[(e^{\rho_1 T} + 1) + \frac{b_1}{\rho_1}(e^{\rho_1 T} - 1) \right] w(j) \end{aligned}$$

for all $j \in S$, where the second inequality is due to Assumption 3.2(ii), and the fourth one follows from Theorem 3.1. Hence, we have

$$\sup_{j \in S} \sup_{s \in [0, T]} \frac{|U^*(i, j, s)|}{w(j)} \leq MT \left[(e^{\rho_1 T} + 1) + \frac{b_1}{\rho_1}(e^{\rho_1 T} - 1) \right] < \infty.$$

Moreover, it follows from the proof of Theorem 4.1 in [23] that there exists a function g on $S \times [0, T]$ satisfying (4.2), and that for each $j \in S$, the partial derivative of g with respect to the second variable s exists, denoted by $\frac{\partial g}{\partial s}$. Thus, we obtain

$$-\frac{\partial g}{\partial s}(j, s) = \inf_{a \in A(j)} \left\{ c_0(j, a) + \sum_{k \in S} g(k, s) q(k|j, a) \right\} \quad \text{for all } j \in S. \quad (4.3)$$

The measurable selection theorem in [13, p.50] and Assumption 3.3 imply that for each $j \in S$, the function $\frac{\partial g}{\partial s}(j, \cdot)$ on $[0, T]$ is measurable. Set $M^* := \sup_{j \in S} \sup_{s \in [0, T]} \frac{|g(j, s)|}{w(j)}$. Then it follows from (4.3), Assumptions 3.1 and 3.2(ii) that

$$\sup_{j \in S} \sup_{s \in [0, T]} \frac{|\frac{\partial g}{\partial s}(j, s)|}{w^2(j)} \leq M + M^*(\rho_1 + b_1 + 2L).$$

Hence, $\frac{\partial g}{\partial s}$ is a $[0, T]$ -uniformly w^2 -bounded function. Note that $g(j, T) = 0$ for all $j \in S$. Therefore, by Theorem 4.1, we have that for each $j \in S$,

$$\begin{aligned} & E_i^\pi \left[\int_s^T \int_A \sum_{k \in S} \left[\int_t^T \frac{\partial g}{\partial v}(k, v) dv \right] q(k|\xi_t, a) \pi(da|\omega, t) dt \middle| \xi_s = j \right] \\ &= E_i^\pi \left[\int_s^T \frac{\partial g}{\partial t}(\xi_t, t) dt \middle| \xi_s = j \right] - \int_s^T \frac{\partial g}{\partial t}(j, t) dt \\ &= -E_i^\pi \left[\int_s^T \int_A \sum_{k \in S} g(k, t) q(k|\xi_t, a) \pi(da|\omega, t) dt \middle| \xi_s = j \right]. \end{aligned} \quad (4.4)$$

On one hand, by (4.3), we obtain

$$-\int_s^T \frac{\partial g}{\partial t}(\xi_t, t) dt \leq \int_s^T \int_A c_0(\xi_t, a) \pi(da|\omega, t) dt + \int_s^T \int_A \sum_{k \in S} g(k, t) q(k|\xi_t, a) \pi(da|\omega, t) dt.$$

Taking the conditional expectation in the both sides of the last inequality, by (4.4), we have

$$\begin{aligned} & -E_i^\pi \left[\int_s^T \frac{\partial g}{\partial t}(\xi_t, t) dt \middle| \xi_s = j \right] \\ &\leq U(i, j, s, \pi) + E_i^\pi \left[\int_s^T \int_A \sum_{k \in S} g(k, t) q(k|\xi_t, a) \pi(da|\omega, t) dt \middle| \xi_s = j \right] \\ &= U(i, j, s, \pi) + \int_s^T \frac{\partial g}{\partial t}(j, t) dt - E_i^\pi \left[\int_s^T \frac{\partial g}{\partial t}(\xi_t, t) dt \middle| \xi_s = j \right] \\ &= U(i, j, s, \pi) - g(j, s) - E_i^\pi \left[\int_s^T \frac{\partial g}{\partial t}(\xi_t, t) dt \middle| \xi_s = j \right], \end{aligned}$$

which implies $g(j, s) \leq U(i, j, s, \pi)$ for all $j \in S$. By the arbitrariness of π , we obtain

$$g(j, s) \leq U^*(i, j, s) \quad \text{for all } j \in S \text{ and } s \in [0, T]. \quad (4.5)$$

On the other hand, it follows from Assumption 3.3 and the measurable selection theorem in [13, p.50] that there exists a measurable function f^* on $S \times [0, T]$ satisfying $f^*(j, s) \in A(j)$ and

$$-\frac{\partial g}{\partial s}(j, s) = c_0(i, f^*(j, s)) + \sum_{k \in S} g(k, s) q(k|j, f^*(j, s))$$

for all $j \in S$ and $s \in [0, T]$. Let $\pi^*(\cdot | \omega, s) := \delta_{f^*(\xi_{s-}(\omega), s)}(\cdot)$. Then, combining the last equality and following the similar arguments of (4.5), we have

$$g(j, s) = U(i, j, s, \pi^*) \geq U^*(i, j, s) \text{ for all } j \in S \text{ and } s \in [0, T]. \quad (4.6)$$

Hence, the statement follows from (4.5) and (4.6).

(b) Part (b) follows directly from the proof of part (a). \square

Remark 4.1. (a) Theorem 4.2 indicates that there exists a finite-horizon optimal deterministic Markov policy in the class of all randomized history-dependent policies.

(b) The existence of a solution to the equality (4.2), the so-called optimality equation for finite-horizon criteria, has been established in [2, 3, 6, 18, 20, 23, 24]. More precisely, the transition rates are assumed to be bounded in [2, 6, 18, 20, 24], and unbounded in [3, 23]. However, the fact that the finite-horizon optimal value function is the unique solution to the optimality equation and the existence of optimal policies have not been studied in [3]. The discussions on the existence of optimal policies are restricted to the class of all randomized Markov policies in the aforementioned works. Theorem 4.2 deals with the finite-horizon criteria in the class of all randomized history-dependent policies, and extends the results in the previous literature. It should be noted that the controlled state process does not have the Markov property under any randomized history-dependent policy, the extension is nontrivial. Moreover, the fixed point theorem and uniformization techniques are inapplicable to the case of unbounded transition rates.

5 Linear programming for the constrained case

In this section we will use the convex analytic approach by introducing the occupation measures of the finite-horizon criteria to deal with the constrained case.

Let w be as in Assumption 3.1, and $X := [0, T] \times K$ is endowed with Borel σ -algebra $\mathcal{B}(X)$. $B_w(X)$ denotes the set of real-valued measurable functions on X with finite norm $\|g\|_w := \sup_{t \in [0, T]} \sup_{(i, a) \in K} \frac{|g(t, i, a)|}{w(i)}$. Denote by $C_w(X)$ the set of all continuous functions in $B_w(X)$, and by $\mathcal{P}_w(X)$ the set of all probability measures η on $\mathcal{B}(X)$ satisfying $\sum_{i \in S} w(i) \bar{\eta}(i) < \infty$, where $\bar{\eta}(i) := \eta([0, T], \{i\}, A)$ for all $i \in S$. The set $\mathcal{P}_w(X)$ is endowed with the w -weak topology for which all mappings $\eta \mapsto \int_{[0, T]} \sum_{i \in S} \int_A g(t, i, a) \eta(dt, i, da)$ are continuous for

each $g \in C_w(X)$. Moreover, because X is metrizable, it follows from Corollary A.44 in [5] that $\mathcal{P}_w(X)$ is metrizable with respect to the w -weak topology.

For each $\pi \in \Pi$, we define the occupation measure of finite-horizon criteria on $\mathcal{B}(X)$ corresponding to π by

$$\eta^\pi(dt, i, da) := \frac{1}{T} E_\gamma^\pi [I_{\{\xi_t=i\}} \pi(da|\omega, t)] dt$$

for any $i \in S$. The set of all occupation measures is denoted by \mathcal{D} , i.e., $\mathcal{D} := \{\eta^\pi : \pi \in \Pi\}$. Moreover, define

$$\mathcal{D}_0 := \left\{ \eta \in \mathcal{D} : T \int_{[0,T]} \sum_{i \in S} \int_A c_n(i, a) \eta(dt, i, a) \leq d_n \text{ for all } 1 \leq n \leq N \right\}. \quad (5.1)$$

Before investigating the properties of \mathcal{D} , we impose the following condition.

Assumption 5.1. (i) For each integer $m \geq 1$, the set $S_m := \{i \in S : w(i) \leq m\}$ is nonempty and finite, where w is as in Assumption 3.1.

(ii) $\gamma(w^2) := \sum_{i \in S} w^2(i) \gamma(i) < \infty$ and $\gamma(w^3) := \sum_{i \in S} w^3(i) \gamma(i) < \infty$.

Remark 5.1. (a) If S is assumed to be the set of all nonnegative integers and the function w on S satisfies $\lim_{i \rightarrow \infty} w(i) = \infty$, Assumption 5.1(i) holds.

(b) Assumption 5.1(i) is used to obtain the compactness of the set of all occupation measures in the w -weak topology.

Under Assumptions 3.1, 3.2(i), 3.3(i), and 5.1(ii), Proposition 2.1 in [19] gives

$$\int_0^T \sum_{i \in S} \int_A w^2(i) \eta^\pi(dt, i, da) = \frac{1}{T} \int_0^T E_\gamma^\pi[w^2(\xi_t)] dt \leq e^{\rho_2 T} \gamma(w^2) + \frac{b_2}{\rho_2} (e^{\rho_2 T} - 1) < \infty \quad (5.2)$$

for all $\pi \in \Pi$. Hence, we have $\mathcal{D}_0 \subseteq \mathcal{D} \subseteq \mathcal{P}_{w^2}(X)$.

Now we give the properties of the occupation measures of finite-horizon criteria below.

Theorem 5.1. Suppose that Assumptions 3.1-3.3, and 5.1 hold. Then we have

(a) If $\eta \in \mathcal{P}_{w^2}(X)$, then $\eta \in \mathcal{D}$ if and only if

$$\begin{aligned} & T \int_{[0,T]} \sum_{i \in S} \int_A \sum_{j \in S} \int_t^T g(j, v) dv q(j|i, a) \eta(dt, i, da) \\ &= T \int_{[0,T]} \sum_{i \in S} g(i, t) \eta(dt, i, A) - \int_{[0,T]} \sum_{i \in S} g(i, t) \gamma(i) dt \end{aligned} \quad (5.3)$$

for each $g \in C_w(S \times [0, T])$.

(b) The sets \mathcal{D} and \mathcal{D}_0 are convex and compact in the w^2 -weak topology.

Proof. (a) If $\eta \in \mathcal{D}$, by the definition of \mathcal{D} , there exists $\pi \in \Pi$ such that $\eta = \eta^\pi$. Fix any $g \in C_w(S \times [0, T])$. Since the function g is $[0, T]$ -uniformly w^2 -bounded, it follows from Theorem 4.1 that

$$\begin{aligned} & T \int_0^T \sum_{i \in S} \int_A \sum_{j \in S} \int_t^T g(j, v) dv q(j|i, a) \eta^\pi(dt, i, da) \\ &= E_\gamma^\pi \left[\int_0^T \int_A \sum_{j \in S} \int_t^T g(j, v) dv q(j|\xi_t, a) \pi(da|\omega, t) dt \right] \\ &= E_\gamma^\pi \left[\int_0^T g(\xi_t, t) dt \right] - \sum_{i \in S} \int_0^T g(i, t) dt \gamma(i) \\ &= T \int_0^T \sum_{i \in S} g(i, t) \eta^\pi(dt, i, A) - \int_0^T \sum_{i \in S} g(i, t) \gamma(i) dt. \end{aligned}$$

Hence, η satisfies (5.3). Conversely, suppose that (5.3) holds for some $\eta \in \mathcal{P}_{w^2}(X)$. By Proposition D.8 in [12, p.184], there exists a kernel φ on A given $S \times [0, T]$ satisfying $\varphi(A(i)|i, t) = 1$ for all $(i, t) \in S \times [0, T]$, and $\eta(dt, i, da) = \eta(dt, i, A) \varphi(da|i, t)$. Let $\pi^*(\cdot | \omega, t) := \varphi(\cdot | \xi_{t-}(\omega), t)$. Below we will show that $\eta = \eta^{\pi^*}$. This is equivalent to proving

$$\int_{[0, T]} \sum_{i \in S} \int_A h(t, i, a) \eta(dt, i, da) = \int_{[0, T]} \sum_{i \in S} \int_A h(t, i, a) \eta^{\pi^*}(dt, i, da) \quad (5.4)$$

for any bounded measurable function h on X . Fix any $j, k \in S$, $t \in [0, T]$, and $h \in B(X)$. Define

$$H(j, t) := E_k^{\pi^*} \left[\int_t^T \int_A h(s, \xi_s, a) \varphi(da|\xi_s, s) ds \middle| \xi_t = j \right].$$

Then, by Theorem 3.2, we obtain

$$\begin{aligned} H(j, t) &= \int_t^T \sum_{i \in S} \int_A h(s, i, a) \varphi(da|i, s) P_k^{\pi^*}(\xi_s = i | \xi_t = j) ds = \int_t^T \int_A h(s, j, a) \varphi(da|j, s) ds \\ &+ \int_t^T \sum_{i \in S} \int_A h(s, i, a) \varphi(da|i, s) \int_t^s \sum_{i' \in S} \int_A q(i|i', a) \varphi(da|i', v) P_k^{\pi^*}(\xi_v = i' | \xi_t = j) dv ds. \end{aligned} \quad (5.5)$$

Let

$$\begin{aligned} \mathcal{L} &:= \left\{ g \in B(S \times [t, T]) : \int_t^T \sum_{i \in S} g(i, s) \int_t^s \sum_{i' \in S} \int_A q(i|i', a) \varphi(da|i', v) P_k^{\pi^*}(\xi_v = i' | \xi_t = j) dv ds \right. \\ &\quad \left. = \int_t^T \int_A \sum_{i' \in S} \int_s^T \sum_{i \in S} g(i, v) P_k^{\pi^*}(\xi_v = i | \xi_s = i') dv q(i'|j, a) \varphi(da|j, s) ds \right\}. \end{aligned}$$

Thus, employing the Kolmogorov forward and backward equations and following the similar arguments of Theorem 4.1, we have $\mathcal{L} = B(S \times [t, T])$. Since the function $\int_A h(s, i, a) \varphi(da|i, s)$ is in $B(S \times [t, T])$, we get

$$\int_t^T \sum_{i \in S} \int_A h(s, i, a) \varphi(da|i, s) \int_t^s \sum_{i' \in S} \int_A q(j|i', a) \varphi(da|i', v) P_k^{\pi^*}(\xi_v = i' | \xi_t = j) dv ds$$

$$= \int_t^T \int_A \sum_{i' \in S} H(i', s) q(i|j, a) \varphi(da|j, s) ds.$$

Hence, combining the last equality and (5.5), we have

$$H(j, t) = \int_t^T \int_A h(s, j, a) \varphi(da|j, s) ds + \int_t^T \int_A \sum_{i \in S} H(i, s) q(i|j, a) \varphi(da|j, s) ds,$$

which implies

$$-\frac{\partial H}{\partial t}(j, t) = \int_A h(t, j, a) \varphi(da|j, t) + \int_A \sum_{i \in S} H(i, t) q(i|j, a) \varphi(da|j, t) \quad \text{a.e. } t \in [0, T]. \quad (5.6)$$

Let $\widetilde{M} := \sup_{(t, j, a) \in X} |h(t, j, a)|$. Then it follows from (5.6) and Assumption 3.1 that

$$\left| \frac{\partial H}{\partial t}(j, t) \right| \leq (1 + 2L) \widetilde{M} w(j) \quad \text{for all } (j, t) \in S \times [0, T]. \quad (5.7)$$

Set $g(i, t) := \widetilde{g}(t)$ for all $(i, t) \in S \times [0, T]$ in (5.3), where \widetilde{g} is an arbitrary bounded measurable function on $[0, T]$. Then we conclude that the marginal of η on $[0, T]$ is the normalized Lebesgue measure. Hence, by Proposition D.8 in [12, p.184], there exists a kernel ϕ on S given $[0, T]$ satisfying $\eta(dt, i, A) = \frac{1}{T} \phi(i, t) dt$. Applying the standard technique, we see that (5.3) is also true for any function in $B_w(S \times [0, T])$. Therefore, direct calculations give

$$\begin{aligned} & \int_{[0, T]} \sum_{i \in S} \int_A h(t, i, a) \eta(dt, i, da) \\ &= \frac{1}{T} \int_{[0, T]} \sum_{i \in S} \int_A h(t, i, a) \varphi(da|i, t) \phi(i, t) dt \\ &= -\frac{1}{T} \int_{[0, T]} \sum_{i \in S} \left[\frac{\partial H}{\partial t}(i, t) + \int_A \sum_{i' \in S} H(i', t) q(i'|i, a) \varphi(da|i, t) \right] \phi(i, t) dt \\ &= -\int_{[0, T]} \sum_{i \in S} \frac{\partial H}{\partial t}(i, t) \eta(dt, i, A) + \int_{[0, T]} \sum_{i \in S} \int_A \sum_{i' \in S} \int_t^T \frac{\partial H}{\partial v}(i', v) dv q(i'|j, a) \eta(dt, i, da) \\ &= -\frac{1}{T} \int_{[0, T]} \sum_{i \in S} \frac{\partial H}{\partial t}(i, t) \gamma(i) dt \\ &= \frac{1}{T} \sum_{i \in S} H(i, 0) \gamma(i), \end{aligned}$$

where the second equality follows from (5.6), and the fourth one is due to (5.3) and (5.7). Moreover, it is obvious that

$$\int_{[0, T]} \sum_{i \in S} \int_A h(t, i, a) \eta^{\pi^*}(dt, i, da) = \frac{1}{T} \sum_{i \in S} H(i, 0) \gamma(i).$$

Thus, the equality (5.4) holds. Hence, we obtain $\eta \in \mathcal{D}$.

(b) The convexity of \mathcal{D} and \mathcal{D}_0 follows directly from part (a) and (5.1). Let $\{\eta_n\} \subseteq \mathcal{D}$ be an arbitrary sequence converging to a measure $\eta \in \mathcal{P}_{w^2}(X)$ in the w^2 -weak topology. Then it follows from part (a) that

$$\begin{aligned} & T \int_0^T \sum_{i \in S} \int_A \sum_{j \in S} \int_t^T g(j, v) dv q(j|i, a) \eta_n(dt, i, da) \\ &= T \int_0^T \sum_{i \in S} g(i, t) \eta_n(dt, i, A) - \int_0^T \sum_{i \in S} g(i, t) \gamma(i) dt \end{aligned} \quad (5.8)$$

for each $g \in C_w(S \times [0, T])$. By Assumptions 3.1 and 3.3(iii), we have that the function $\sum_{j \in S} \int_t^T g(j, v) dv q(j|i, a)$ belongs to $C_{w^2}(X)$. Thus, combining (5.8) and part (a), we obtain $\eta \in \mathcal{D}$. Hence, \mathcal{D} is closed in the w^2 -weak topology.

For each integer $m \geq 1$, define $X_m := \{(t, i, a) : t \in [0, T], i \in S_m, a \in A(i)\}$ and $w_m := \inf\{w(i) : (t, i, a) \notin X_m\}$, where S_m is as in Assumption 5.1. Then it follows from Assumption 3.3(ii) that $\{X_m\}$ is a nondecreasing sequence of compact sets, $X_m \uparrow X$, and $\lim_{m \rightarrow \infty} w_m = \infty$. Thus, we have

$$w_m \int_0^T \sum_{i \notin S_m} \int_A w^2(i) \eta^\pi(dt, i, da) \leq \frac{1}{T} \int_0^T E_\gamma^\pi[w^3(\xi_t)] dt \leq e^{\rho_3 T} \gamma(w^3) + \frac{b_3}{\rho_3} (e^{\rho_3 T} - 1)$$

for all $\pi \in \Pi$. Hence, for any $\epsilon > 0$, the last inequality implies that there exists an integer $m_0 > 0$ satisfying

$$\sup_{\pi \in \Pi} \int_0^T \sum_{i \notin S_{m_0}} \int_A w^2(i) \eta^\pi(dt, i, da) \leq \epsilon. \quad (5.9)$$

Therefore, by (5.2), (5.9), and Corollary A.46 in [5, p.424], we have that \mathcal{D} is compact in the w^2 -weak topology. Finally, it follows from the compactness of \mathcal{D} , (5.1), Assumptions 3.2(ii) and 3.3(iii) that \mathcal{D}_0 is also compact in the w^2 -weak topology. \square

Remark 5.2. (a) Theorem 5.1 is *new*, and it gives an equivalent characterization of the occupation measures and establishes the compactness and convexity of the set of all occupation measures, which play a crucial role in reformulating the constrained optimization as a linear program and obtaining its dual program.

(b) From the proof of part (a), we conclude that for each occupation measure generated by a randomized history-dependent policy, there exists an occupation measure generated by a randomized Markov policy equal to it. Hence, for any $\pi \in \Pi$, there exists $\tilde{\pi} \in \Pi^M$ such that $V_n(\pi) = V_n(\tilde{\pi})$ for all $1 \leq n \leq N$.

The constrained optimization problem (2.3) is equivalent to the linear programming formulation below:

$$\text{minimize} \quad T \int_0^T \sum_{i \in S} \int_A c_0(i, a) \eta(dt, i, da) \text{ over } \eta \in \mathcal{D}$$

$$\text{subject to } T \int_0^T \sum_{i \in S} \int_A c_n(i, a) \eta(dt, i, da) \leq d_n, \quad n = 1, 2, \dots, N. \quad (5.10)$$

The following statement establishes the existence of constrained-optimal policies for the case of unbounded transition and cost rates.

Theorem 5.2. *Suppose that Assumptions 3.1-3.3 and 5.1 are satisfied. Then there exists a constrained-optimal policy $\pi^* \in \Pi^M$ for the constrained optimization problem (2.3).*

Proof. It follows from the proof of part (a) in Theorem 5.1 that $\mathcal{D} = \{\eta^\pi : \pi \in \Pi^M\}$. Hence, the desired assertion follows from (5.10), Assumptions 3.2(ii), 3.3(iii), Theorem 5.1, and the Weierstrass theorem in [1, p.40]. \square

Below we will develop the dual program of the linear program (5.10), and provide the strong duality conditions. To this end, we introduce the following notation.

Let w be as in Assumption 3.1, and $Y := [0, T] \times S$ is endowed with Borel σ -algebra $\mathcal{B}(Y)$. We denote by \mathbb{R} the space of all real numbers (i.e., $\mathbb{R} := (-\infty, \infty)$), by $\mathcal{M}_{w^3}(X)$ the space of all signed measures η on $\mathcal{B}(X)$ with $\int_{[0, T]} \sum_{i \in S} \int_A w^3(i) |\eta|(dt, i, da) < \infty$, and by $\mathcal{M}_{w^3}^+(X) := \{\eta \in \mathcal{M}_{w^3}(X) : \eta \geq 0\}$, where $|\eta| := \eta^+ + \eta^-$ denotes the total variation of η . $\mathcal{M}_{w^2}(Y)$ and $\mathcal{M}_{w^2}^+(Y)$ are defined similarly. Moreover, let

$$\begin{aligned} \mathcal{X} &:= \mathcal{M}_{w^3}(X) \times \mathbb{R}^N, \quad \mathcal{Y} := B_{w^3}(X) \times \mathbb{R}^N, \\ \mathcal{Z} &:= \mathcal{M}_{w^2}(Y) \times \mathbb{R}^N, \quad \mathcal{U} := B_{w^2}(Y) \times \mathbb{R}^N. \end{aligned}$$

Define a bilinear map $\langle \rangle_1$ on the dual pair of $(\mathcal{X}, \mathcal{Y})$ by

$$\langle (\eta, x_1, \dots, x_N), (g, y_1, \dots, y_N) \rangle_1 := \int_{[0, T]} \sum_{i \in S} \int_A g(t, i, a) \eta(dt, i, da) + \sum_{n=1}^N x_n y_n \quad (5.11)$$

for all $(\eta, x_1, \dots, x_N) \in \mathcal{X}$ and $(g, y_1, \dots, y_N) \in \mathcal{Y}$, and another bilinear $\langle \rangle_2$ on the dual pair of $(\mathcal{Z}, \mathcal{U})$ by

$$\langle (\nu, z_1, \dots, z_N), (h, u_1, \dots, u_N) \rangle_2 := \int_{[0, T]} \sum_{i \in S} h(t, i) \nu(dt, i) + \sum_{n=1}^N z_n u_n \quad (5.12)$$

for all $(\nu, z_1, \dots, z_N) \in \mathcal{Z}$ and $(h, u_1, \dots, u_N) \in \mathcal{U}$. Moreover, two operators Γ from \mathcal{X} to \mathcal{Z} , and Γ^* from \mathcal{U} to \mathcal{Y} are defined as follows:

$$\begin{aligned} \Gamma(\eta, x_1, \dots, x_N) &:= \left(\eta(ds, j, A) - \int_{[0, T]} \sum_{i \in S} \int_A I_{[t, T]}(s) q(j|i, a) \eta(dt, i, da) ds, \right. \\ &\quad \left. T \int_{[0, T]} \sum_{i \in S} \int_A c_1(i, a) \eta(dt, i, da) + x_1, \dots, T \int_{[0, T]} \sum_{i \in S} \int_A c_N(i, a) \eta(dt, i, da) + x_N \right), \end{aligned}$$

and

$$\Gamma^*(h, u_1, \dots, u_N) := \left(h(t, i) - \int_t^T \sum_{j \in S} h(s, j) q(j|i, a) ds + T \sum_{n=1}^N u_n c_n(i, a), u_1, \dots, u_N \right).$$

Then we have the following lemma on the properties of Γ and Γ^* .

Lemma 5.1. *Under Assumptions 3.1-3.3, and 5.1, the following statements hold.*

- (a) $\Gamma\mathcal{X} \subseteq \mathcal{Z}$ and $\Gamma^*(\mathcal{U}) \subseteq \mathcal{Y}$.
- (b) Γ^* is the adjoint of Γ .
- (c) Γ is $\tau(\mathcal{X}, \mathcal{Y}) - \tau(\mathcal{Z}, \mathcal{U})$ continuous, where $\tau(\mathcal{X}, \mathcal{Y})$ denotes the coarsest topology on \mathcal{X} such that $\langle \cdot, y \rangle_1$ is continuous on \mathcal{X} for each $y \in \mathcal{Y}$, and $\tau(\mathcal{Z}, \mathcal{U})$ is defined similarly.

Proof. (a) For each $(\eta, x_1, \dots, x_N) \in \mathcal{X}$, it follows from Assumption 3.2(ii) that

$$\int_{[0, T]} \sum_{i \in S} \int_A |c_n(i, a)| |\eta|(dt, i, da) \leq M \int_{[0, T]} \sum_{i \in S} \int_A w(i) |\eta|(dt, i, da) < \infty$$

for all $1 \leq n \leq N$. Moreover, direct calculations yield

$$\begin{aligned} & \int_{[0, T]} \sum_{i \in S} w^2(i) |\eta|(dt, i, A) + \int_{[0, T]} \sum_{j \in S} w^2(j) \int_{[0, T]} \sum_{i \in S} \int_A I_{[t, T]}(s) ds |q(j|i, a)| |\eta|(dt, i, da) \\ & \leq [1 + T(\rho_2 + b_2 + 2L)] \int_{[0, T]} \sum_{i \in S} w^3(i) |\eta|(dt, i, A) < \infty. \end{aligned}$$

Hence, we obtain $\Gamma\mathcal{X} \subseteq \mathcal{Z}$.

For each $(h, u_1, \dots, u_N) \in \mathcal{U}$, set $\bar{L} := \sup_{(t, i) \in Y} \frac{|h(t, i)|}{w^2(i)}$. Then, by Assumptions 3.1(ii), 3.2(ii), and 3.3(i), we have

$$\begin{aligned} & \sup_{(t, i, a) \in X} \frac{\left| h(t, i) - \int_t^T \sum_{j \in S} h(s, j) q(j|i, a) ds + \sum_{n=1}^N u_n c_n(i, a) \right|}{w^3(i)} \\ & \leq \bar{L}[1 + T(\rho_2 + b_2 + 2L)] + M \sum_{n=1}^N |u_n|, \end{aligned}$$

which implies $\Gamma^*(\mathcal{U}) \subseteq \mathcal{Y}$.

(b) For each $(\eta, x_1, \dots, x_N) \in \mathcal{X}$ and $(h, u_1, \dots, u_N) \in \mathcal{U}$, it follows from (5.11), (5.12), and the definitions of Γ and Γ^* that

$$\begin{aligned} & \langle \Gamma(\eta, x_1, \dots, x_N), (h, u_1, \dots, u_N) \rangle_2 \\ & = \int_{[0, T]} \sum_{i \in S} h(t, i) \eta(dt, i, A) - \int_{[0, T]} \sum_{j \in S} h(s, j) \int_{[0, T]} \sum_{i \in S} \int_A I_{[t, T]}(s) ds q(j|i, a) \eta(dt, i, da) \end{aligned}$$

$$\begin{aligned}
& + \sum_{n=1}^N \left[T \int_{[0,T]} \sum_{i \in S} \int_A c_n(i, a) \eta(dt, i, da) + x_n \right] u_n \\
& = \int_{[0,T]} \sum_{i \in S} h(t, i) \eta(dt, i, A) - \int_{[0,T]} \sum_{i \in S} \int_A \int_t^T \sum_{j \in S} h(s, j) q(j|i, a) ds \eta(dt, i, da) \\
& \quad + \sum_{n=1}^N T u_n \int_{[0,T]} \sum_{i \in S} \int_A c_n(i, a) \eta(dt, i, da) + \sum_{n=1}^N x_n u_n \\
& = \langle (\eta, x_1, \dots, x_N), \Gamma^*(h, u_1, \dots, u_N) \rangle_1.
\end{aligned}$$

Therefore, we get the desired assertion.

(c) Part (c) follows from part (a) and Proposition 12.2.5 in [13, p.208]. \square

According to Lemma 5.1 and Chapter 12 in [13], the constrained problem (5.10) can be rewritten as

$$\begin{aligned}
\mathbf{P} : \quad & \text{minimize} \quad \langle (\eta, x_1, \dots, x_N), (Tc_0, 0, \dots, 0) \rangle_1 \\
& \text{subject to} \quad \Gamma(\eta, x_1, \dots, x_N) = \left(\frac{1}{T} \gamma dt, d_1, \dots, d_N \right) \\
& \quad \eta \in \mathcal{M}_{w^3}^+(X), \quad x_1 \geq 0, \dots, x_N \geq 0.
\end{aligned} \tag{5.13}$$

The corresponding dual problem of \mathbf{P} is

$$\begin{aligned}
\mathbf{P}^* : \quad & \text{maximize} \quad \langle \left(\frac{1}{T} \gamma dt, d_1, \dots, d_N \right), (h, u_1, \dots, u_N) \rangle_2 \\
& \text{subject to} \quad Tc_0(i, a) - h(t, i) + \int_t^T \sum_{j \in S} h(s, j) q(j|i, a) ds - T \sum_{n=1}^N u_n c_n(i, a) \geq 0 \\
& \quad \text{for all } (t, i, a) \in X, \quad h \in B_{w^2}(Y), \quad u_1 \leq 0, \dots, u_N \leq 0.
\end{aligned} \tag{5.14}$$

We denote the values of problems (5.13) and (5.14) by $\inf(\mathbf{P})$ and $\sup(\mathbf{P}^*)$, respectively.

In order to establish the strong duality between the primal linear program (5.13) and its dual program (5.14), we impose the following Slater condition which is commonly used in the constrained optimization problems; see, for instance, [8, 10, 11, 19].

Assumption 5.2. *There exists a policy $\tilde{\pi} \in \Pi$ such that $V_n(\tilde{\pi}) < d_n$ for all $1 \leq n \leq N$.*

Now we present the strong duality theorem for finite-horizon criteria below.

Theorem 5.3. *Suppose that Assumptions 3.1-3.3, 5.1, and 5.2 hold. Then problems (5.13) and (5.14) admit optimal solutions, and $\inf(\mathbf{P}) = \sup(\mathbf{P}^*)$.*

Proof. By Theorem 5.2, we obtain that the problem (5.13) admits an optimal solution. Below we will show the existence of optimal solutions for the problem (5.14). It follows from

Theorem 5.1 that \mathcal{D} is convex. By Assumption 5.2, Theorem 17 and Example 1 in [22, p.7, 18, 23], there exist constants $u_n^* \geq 0$ ($1 \leq n \leq N$) such that

$$\begin{aligned} \inf(\mathbf{P}) &= \sup_{u_n \geq 0, 1 \leq n \leq N} \inf_{\eta \in \mathcal{D}} \left\{ T \int_{[0,T]} \sum_{i \in S} \int_A \left(c_0(i, a) + \sum_{n=1}^N u_n c_n(i, a) \right) \eta(dt, i, da) - \sum_{n=1}^N u_n d_n \right\} \\ &= \inf_{\eta \in \mathcal{D}} \left\{ T \int_{[0,T]} \sum_{i \in S} \int_A \left(c_0(i, a) + \sum_{n=1}^N u_n^* c_n(i, a) \right) \eta(dt, i, da) - \sum_{n=1}^N u_n^* d_n \right\}. \end{aligned} \quad (5.15)$$

Moreover, when $c_0(i, a)$ in (4.2) is replaced by $T(c_0(i, a) + \sum_{n=1}^N u_n^* c_n(i, a)) - \sum_{n=1}^N u_n^* d_n$, following the similar arguments of Theorem 4.2, we conclude that there exists a function $h^* \in B_w(Y)$ such that for each $i \in S$ and $t \in [0, T]$,

$$h^*(t, i) = \int_t^T \inf_{a \in A(i)} \left\{ T c_0(i, a) + T \sum_{n=1}^N u_n^* c_n(i, a) - \sum_{n=1}^N u_n^* d_n + \sum_{j \in S} h^*(s, j) q(j|i, a) \right\} ds,$$

which implies

$$-\frac{\partial h^*}{\partial t}(t, i) = \inf_{a \in A(i)} \left\{ T c_0(i, a) + T \sum_{n=1}^N u_n^* c_n(i, a) - \sum_{n=1}^N u_n^* d_n + \sum_{j \in S} h^*(t, j) q(j|i, a) \right\}. \quad (5.16)$$

Let $\tilde{h}(t, i) := \sum_{n=1}^N u_n^* d_n - \frac{\partial h^*}{\partial t}(t, i)$ for all $(t, i) \in Y$. Then, by (5.16), we have

$$T c_0(i, a) - T \sum_{n=1}^N (-u_n^*) c_n(i, a) - \tilde{h}(t, i) + \int_t^T \sum_{j \in S} \tilde{h}(s, j) q(j|i, a) ds \geq 0$$

for all $(t, i, a) \in X$. As in the proof of Theorem 4.2, we obtain that $\tilde{h} \in B_{w^2}(Y)$, and

$$\begin{aligned} \sum_{i \in S} h^*(0, i) \gamma(i) &= \inf_{\pi \in \Pi} E_\gamma^\pi \left[\int_0^T \int_A \left(T c_0(\xi_t, a) + T \sum_{n=1}^N u_n^* c_n(\xi_t, a) - \sum_{n=1}^N u_n^* d_n \right) \pi(da|\omega, t) dt \right] \\ &= T \inf(\mathbf{P}), \end{aligned} \quad (5.17)$$

where the second equality is due to (5.15). Hence, $(\tilde{h}, -u_1^*, \dots, -u_N^*)$ is feasible for \mathbf{P}^* . Direct calculations give

$$\begin{aligned} \sup(\mathbf{P}^*) &\geq \langle (\frac{1}{T} \gamma dt, d_1, \dots, d_N), (\tilde{h}, -u_1^*, \dots, -u_N^*) \rangle_2 \\ &= \frac{1}{T} \int_0^T \sum_{i \in S} \tilde{h}(t, i) \gamma(i) dt - \sum_{n=1}^N u_n^* d_n \\ &= -\frac{1}{T} \int_0^T \sum_{i \in S} \frac{\partial h^*}{\partial t}(t, i) \gamma(i) dt \\ &= \frac{1}{T} \sum_{i \in S} h^*(0, i) \gamma(i) = \inf(\mathbf{P}). \end{aligned} \quad (5.18)$$

where the last equality follows from (5.17). Moreover, by Theorem 12.2.9 in [13, p.213] and Theorem 5.2, we have $\sup(\mathbf{P}^*) \leq \inf(\mathbf{P})$. Hence, this inequality and (5.18) gives $\inf(\mathbf{P}) = \sup(\mathbf{P}^*)$, which implies that $(\tilde{h}, -u_1^*, \dots, -u_N^*)$ is an optimal solution for \mathbf{P}^* . \square

6 An Example

In this section, we will show the applications of finite-horizon expected total cost criteria with a controlled birth and death system, which has been used to illustrate the existence of average constrained-optimal policies in [10].

Example 6.1. (A controlled birth and death system in [10]). The control model is given as follows: $S := \{0, 1, 2, \dots\}$, $A(0) := [-\lambda, \lambda] \times \{0\}$, $A(i) := [-\lambda, \lambda] \times [-\mu, \mu]$ for all $i \geq 1$, $q(1|0, (a_1, 0)) = -q(0|0, (a_1, 0)) := \lambda + a_1$ for all $a_1 \in [-\lambda, \lambda]$, and for each $i \geq 1$, $a = (a_1, a_2) \in A(i)$,

$$q(j|i, a) := \begin{cases} \lambda i + a_1, & \text{if } j = i + 1, \\ -(\mu + \lambda)i - a_1 - a_2, & \text{if } j = i, \\ \mu i + a_2, & \text{if } j = i - 1, \\ 0, & \text{otherwise,} \end{cases}$$

where positive constants λ and μ denote the birth and death rates, respectively.

To ensure the existence of optimal policies for the unconstrained and constrained cases, we consider the following conditions.

(C1) There exists a positive constant M such that $|c_n(i, a)| \leq M(i + 1)$ for all $(i, a) \in K$ and $0 \leq n \leq N$.

(C2) For each fixed $i \in S$ and $n \in \{0, 1, \dots, N\}$, $c_n(i, a)$ is continuous in $a \in A(i)$.

(C3) There exists a kernel $\hat{\varphi}$ on A given $S \times [0, T]$ such that $\int_{A(i)} c_n(i, a) \hat{\varphi}(da|i, t) < d_n$ for all $(i, t) \in S \times [0, T]$ and $1 \leq n \leq N$.

(C4) The initial distribution γ on S satisfies $\sum_{i \in S} i^3 \gamma(i) < \infty$.

Proposition 6.1. *Under conditions (C1) and (C2), the controlled birth and death system above satisfies Assumptions 3.1, 3.2(ii), and 3.3. Hence, (by Theorem 4.2), there exists a finite-horizon optimal policy. Moreover, under conditions (C1)-(C4), the controlled birth and death system satisfies Assumptions 3.1-3.3, 5.1, 5.2. Hence, (by Theorem 5.3), there exists a constrained-optimal policy and the strong duality holds.*

Proof. Let $w(i) := i + 1$ for all $i \in S$. For $i = 0$ and $a = (a_1, a_2) \in A(0)$, we have

$$\begin{aligned} q^*(0) &\leq 2\lambda w(0), \\ \sum_{j \in S} w(j) q(j|0, a) &= \lambda + a_1 \leq \lambda w(0) + \lambda, \\ \sum_{j \in S} w^2(j) q(j|0, a) &= 3(\lambda + a_1) \leq 3\lambda w^2(0) + 3\lambda, \end{aligned} \tag{6.1}$$

$$\sum_{j \in S} w^3(j)q(j|0, a) = 7(\lambda + a_1) \leq 7\lambda w^3(0) + 7\lambda.$$

For each $i \geq 1$ and $a = (a_1, a_2) \in A(i)$, straightforward calculations give

$$\begin{aligned} q^*(i) &\leq (\lambda + \mu)i + \lambda + \mu = (\lambda + \mu)w(i), \\ \sum_{j \in S} w(j)q(j|i, a) &= (\lambda - \mu)i + a_1 - a_2 \leq (\lambda + \mu)w(i), \\ \sum_{j \in S} w^2(j)q(j|i, a) &= 2(\lambda - \mu)i^2 + (3\lambda - \mu + 2a_1 - 2a_2)i + 3a_1 - a_2 \\ &\leq (7\lambda + 5\mu)w^2(i) + 3\lambda + \mu, \\ \sum_{j \in S} w^3(j)q(j|i, a) &= 3(\lambda - \mu)i^3 + (9\lambda - 3\mu + 3a_1 - 3a_2)i^2 \\ &\quad + (7\lambda - \mu + 9a_1 - 3a_2)i + 7a_1 - a_2 \\ &\leq (31\lambda + 13\mu)w^3(i) + 7\lambda + \mu. \end{aligned} \tag{6.2}$$

Hence, Assumptions 3.1 and 3.3(i) are satisfied with $L := 2\lambda + \mu$, $\rho_1 := \lambda + \mu$, $b_1 := \lambda$, $\rho_2 := 7\lambda + 5\mu$, $b_2 := 3\lambda + \mu$, $\rho_3 := 31\lambda + 13\mu$, and $b_3 := 7\lambda + \mu$. Moreover, by the description of the model, (6.1), (6.2), and conditions (C1)-(C4), we see that Assumptions 3.2, 3.3(ii)(iii), 5.1(ii), and 5.2 hold. Finally, for each integer $m \geq 1$, we obtain $S_m = \{0, 1, \dots, m-1\}$ which implies Assumption 5.1(i). This completes the proof of the proposition. \square

Remark 6.1. (a) Since the finite-horizon optimality criterion is different from the average optimality criterion discussed in [10], the conditions used in Example 6.1 also differ from those in [10]. For example, the condition “ $\lambda > \mu$ ” in [10] is not required for Example 6.1, whereas there is no need to impose condition (C4) in [10].

(b) The transition rates in [2, 6, 20, 24] are assumed to be bounded. Therefore, the conditions in the aforementioned works are inapplicable to Example 6.1 because it allows the transition rates to be unbounded from above and from below.

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